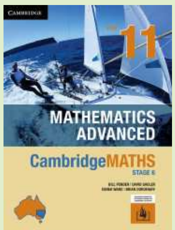
NORMANHURST BOYS HIGH SCHOOL

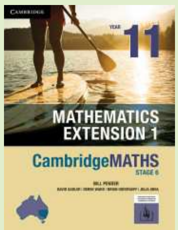
MATHEMATICS ADVANCED INCORPORATING EXTENSION 1 (YEAR 11 COURSE)

Topic summary and exercises:

Ⓐ ⓍⅠ **Probability**

With references to





Name: ....

Initial version by H. Lam, October 2019. Last updated September 4, 2020.
Various corrections by students & members of the Department of Mathematics at Normanhurst Boys High School.

Acknowledgements Pictograms in this document are a derivative of the work originally by Freepik at `http://www.flaticon.com`, used under 🅭 CC BY 2.0.

Symbols used

- (!) Beware! Heed warning.
- (F) Provided on NESA Reference Sheet
- (M) Facts/formulae to memorise.
- (A) Mathematics Advanced content.
- (X1) Mathematics Extension 1 content.
- (L) Literacy: note new word/phrase.
- (N) Further reading/exercises to enrich your understanding and application of this topic.
- (U) Facts/formulae to understand, as opposed to blatant memorisation.

$\mathbb{N}$ the set of natural numbers

$\mathbb{Z}$ the set of integers

$\mathbb{Q}$ the set of rational numbers

$\mathbb{R}$ the set of real numbers

$\forall$ for all

Syllabus outcomes addressed

MA11-7 uses concepts and techniques from probability to present and interpret data and solve problems in a variety of contexts, including the use of probability distributions

Syllabus subtopics

MA-S1 (1.1) Probability and Discrete Probability Distributions (Probability and Venn Diagrams)

Gentle reminder

- For a thorough understanding of the topic, *every* question in this handout is to be completed!
- Additional questions from *CambridgeMATHS Year 11 Extension 1* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019b) or *CambridgeMATHS Year 11 Advanced* (Pender, Sadler, Ward, Dorofaeff, & Shea, 2019a) will be completed at the discretion of your teacher.
- Remember to copy the question into your exercise book!

Contents

Section 1

Language of probability

1.1 Definitions

Learning Goal(s)

Knowledge
Probability terminology

Skills
Read terminology associated with probability

Understanding
Boundary conditions for probability

By the end of this section am I able to:

13.1 Understand and use the concepts and language associated with theoretical probability, relative frequency and the probability scale.

13.2 Understand the definition of complementary events, and the result $P(\overline{E}) = 1 - P(E)$

13.3 Understand the definition of experimental probability as a measure of relative frequency.

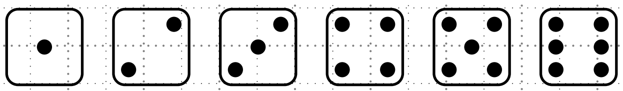
1.1.1 Spaces and sets

Definition 1

Ⓛ The ....sample.... *space* is the set of all possible outcomes of an experiment.

Example When throwing a single six sided die, the sample space is

$S = \{$ ....1, 2, 3, 4, 5, 6........ $\}$

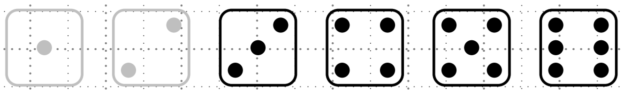


Definition 2

Ⓛ The ....event.... *space* is the set of favourable/desired outcomes for an experiment.

Example The .... event .... space of scoring 'at least 3' when throwing a single six sided die:

$E = \{$ .... 3, 4, 5, 6 .... $\}$



Theorem 1

The ...... event ...... space is always a subset of the ...... sample ...... space, i.e.

$$E \subseteq S$$

Draw a Venn diagram of the above situation.

Definition 3

The *probability* of an event E occurring:

$$P(E) = \frac{|E|}{|S|}$$

where

- $|E|$ is size of the ....event.... space
- $|S|$ is the size of the ....sample.... space

1.1.2 Equally likely outcomes

Definition 4

Ⓛ An experiment has *equally likely outcomes* if one and only one of the n possibilities will occur.

Draw a table of outcomes for roll of one die.

Write a sentence describing an experiment with equally likely outcomes, and draw a table of outcomes.

1.1.3 Impossible and Certain Events

Write a sentence describing an experiment with an impossible event involving rolling one die or drawing a card from a deck of playing cards.

Write a sentence describing an experiment with a certain event involving rolling one dice or drawing a card from a deck of playing cards.

Theorem 2

Conclusion All other probabilities:

$$0 \le P(E) \le 1$$

(This has been a fact that has been taken for granted)

1.1.4 Complementary events

- Often easier to find the probability that an event does not occur than the probability that it does occur

Definition 5

The *complement* of an event E occurring is denoted $\overline{E}$.
(Other texts: E^c, E')

Important note

Ⓛ When the word/phrase *not* or *at least once* appears in problems, it is more than likely channelling you to use the complementary situation, i.e.

$$P(\overline{E}) = 1 - P(E)$$

Be careful with counting - in Extension 1 problems it may not be so straightforward.

Example 1

[2014 2U HSC Q10] Three runners compete in a race. The probabilities that the three runners finish the race in under 10 seconds are $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{2}{5}$, respectively.

What is the probability that at least one of the three runners will finish the race in under 10 seconds?

(A) $\frac{1}{60}$ (B) $\frac{37}{60}$ (C) $\frac{3}{8}$ (D) $\frac{5}{8}$

Important note

The *multiplication principle* $P(A \cap B) = P(A)P(B)$ was used in this example. It applies only to ....independent............ ......event...... .

1.2 Experimental probability

- Also known as the ....relative.... ....frequency.... of an event occurring.

Example 2

A drawing pin was thrown 400 times and fell point-up 362 times.

(a) What were the relative frequencies of the drawing pin falling point-up, and of falling point-down?

(b) What probability does this experiment suggest for the result 'point-up'?

(c) A machine later repeated the experiment 1 000 000 times, and the pin fell point-up 916 203 times.
Does this change your estimate in part (b)?

Example 3

[Ex 12A Q19] Fifty tagged fish were released into a dam known to contain fish. Later, a sample of 30 fish was netted from this dam, of which eight were found to be tagged.

Estimate the total number of fish in the dam just prior to the sample of 30 being removed.

Example 4

[**2011 2U HSC Q1**] A batch of 800 items is examined. The probability that an item from this batch is defective is 0.02.

How many items from this batch are defective?

1.3 Invalid arguments

Consider the following statement:

> *This mathematics class has 24 students. The probability at the end of Year 12 that a student will be Rank 1 in the class is* $\frac{1}{24}$

Identify the fallacy:

- There are 24 possible ranks within this class is correct.
- BUT why is each student ....equally.... ....likely.... to be ranked first?

Further exercises

(A) **Ex 10A**
- Even numbered questions

(X1) **Ex 12A**
- Even numbered questions
- Q21, 22

Section 2

Pictorial representation of sample spaces

Learning Goal(s)

Knowledge
What *arrays*, *dot diagrams* and *tree diagrams* are

Skills
Use these types of diagrams to solve problems

Understanding
When to use each type of diagram

By the end of this section am I able to:

13.4 Use arrays, dot diagrams and tree diagrams to determine the outcomes and probabilities for multi-stage experiments.

Important note

Draw .... picture ......!

2.1 Table

Fill in the spaces

- Use a table when there are a small number of independent events with a small sample space. ($|S| < 36$)
- Usual scenario: two dice, or one die rolled twice)

Example 5

A die is thrown twice. Find the probability that:

(a) the pair is a double

(b) at least one number is four

(c) both numbers are greater than four

(d) both numbers are even

(e) the sum of the two numbers is six

(f) the sum is at most four

Example 6

[2018 2U HSC Q16] A game involves rolling two six-sided dice, followed by rolling a third six-sided die. To win the game, the number rolled on the third die must lie between the two numbers rolled previously. For example, if the first two dice show 1 and 4, the game can only be won by rolling a 2 or 3 with the third die.

(i) What is the probability that a player has no chance of winning before rolling the third die? **2**

(ii) What is the probability that a player wins the game? **2**

Answer: (i) $\frac{4}{9}$ (ii) $\frac{5}{27}$

2.2 Tree diagrams

✎ Fill in the spaces

- Use a tree diagram when there are
  - ....Dependent........ events where after the first stage, the sample space changes.
  - ....Independent........ events with a small number (2-3) outcomes per stage.
- Easy to calculate the ....complementary........ ....situation........ .

Important note

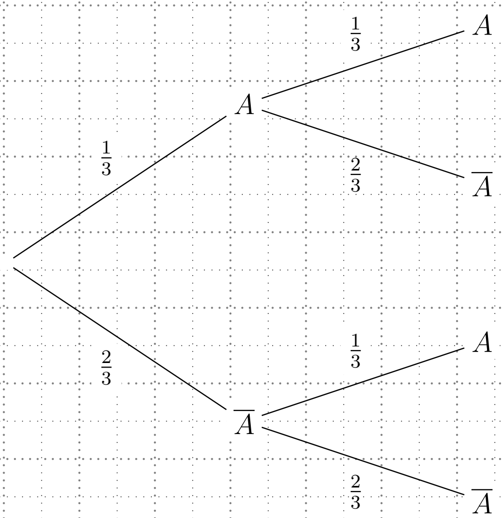
(!) If a stage contains more than ....three.... outcomes, consider how to simplify the tree.

(!) Be careful with ....imbalanced........ trees where some branches of a tree ....terminate........ ....early.... .

(!) Sometimes it may be easier to redraw a tree based on results from previous trees.

Label the tree with the following words/phrases

Outcome	Stage	Probability	Branch



Example 7

[2014 CSSA 2U Q14] Xavier is playing a variation of Chess called Makruk online. Each game is graded and Xavier begins with a 0.6 probability of winning and a 0.3 probability of losing.

At the end of each game, players receive points.

If Xavier wins he receives 5 points. If he loses he receives 2 points and a draw will result in Xavier receiving 3 points.

(i) What is the probability that Xavier's first game ends in a draw? **1**

In the next game, after grading has occurred, Xavier now has a 0.4 probability of winning and a 0.4 probability of losing. Find the probability that after two games:

(ii) Xavier receives ten points **2**

(iii) Xavier receives five points or less **2**

Answer: (i) $\frac{1}{10}$ (ii) $\frac{12}{50}$ (iii) $\frac{11}{50}$

Example 8

[2017 VCE Mathematical Methods Paper 1 Q5] For Jac to log on to a computer successfully, Jac must type the correct password. Unfortunately, Jac has forgotten the password. If Jac types the wrong password, Jac can make another attempt. The probability of success on any attempt is $\frac{2}{5}$. Assume that the result of each attempt is independent of the result of any other attempt. A maximum of three attempts can be made.

(a) What is the probability that Jac does not log on to the computer successfully? **1**

(b) Calculate the probability that Jac logs on to the computer successfully. Express your answer in the form $\frac{a}{b}$, where a and $b \in \mathbb{Z}^+$. **1**

(c) Calculate the probability that Jac logs on to the computer successfully on the second or on the third attempt. Express your answer in the form $\frac{c}{d}$, where c and $d \in \mathbb{Z}^+$. **2**

Answer: (a) $\frac{27}{125}$ (b) $\frac{98}{125}$ (c) $\frac{48}{125}$

Example 9

(!) Jayne has a bag containing 5 strawberry lollies and 3 honey lollies while Laura has another bag containing 4 strawberry lollies and 5 lemon lollies. Each of these two girls is to select at random 3 lollies from their own bag, without replacement.

What is the probability that only one of the two girls will select at least one strawberry lolly from her bag? **Answer:** $\frac{13}{98}$

2.3 Mixed bag

- Some questions may require both tools to be used.

Example 10

[1998 2U HSC Q10] A game is played in which two coloured dice are thrown once. The six faces of the blue die are numbered 4, 6, 8, 9, 10 and 12. The six faces of the pink die are numbered 2, 3, 5, 7, 11 and 13. The player wins if the number on the pink die is larger than the number on the blue die.

(i) By drawing up a table of possible outcomes, or otherwise, calculate the probability of the player winning a game. **3**

(ii) Calculate the probability that the player wins at least once in two successive games. **2**

Answer: (i) $\frac{7}{18}$ (ii) $\frac{203}{324}$

Further exercises

(A) **Ex 10B**
- Even numbered questions

(X1) **Ex 12B**
- Even numbered questions

Section 3

Venn Diagrams

Learning Goal(s)

Knowledge
What *set notation* and *Venn diagrams* are

Skills
Use set notation and Venn diagrams as well as associated rules to solve problems

Understanding
When to use Venn diagrams to solve problems

By the end of this section am I able to:

13.6 Use set notation for describing sets

13.7 Use Venn diagrams, set language and notation

13.8 Understand the counting rule

13.9 Establish and use the rules

$$P(\overline{A}) = 1 - P(A) \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

13.10 Observe that for mutually exclusive events,

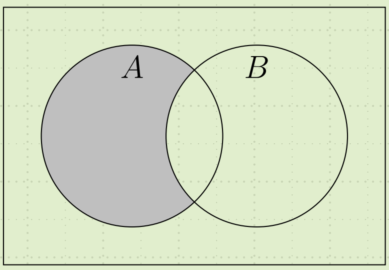
$$P(A \cup B) = P(A) + P(B)$$

History

Image: Wikipedia

Venn diagrams are due to John Venn (1834-1923), is a diagram that shows all possible logical relations between a finite collection of different sets.

$A \cap \overline{B}$



3.1 Set notation and definitions

3.1.1 Symbols used

Definition 6

Set notation (read left-to-right)

- $\in$: *belongs to the* ......set...... ....of....
- $\cap$: *and* (more formally: ....intersection..........)
- $\cup$: *or* (more formally: ......union......)
- $A \subset B$: *A is a* ......subset...... *of* B
- $A \supseteq B$: *A is a* ......superset...... ....or.... ......equal...... ....to.... B.
- $\varnothing$: empty set. A set with no elements in it.

3.1.2 Size (Cardinality) of a set

Definition 7

The *size* of a set A is given the symbol $|A|$.

Example 11

If

- $A = \{\text{all multiples of 3 from 3 to 50 inclusive}\}$
- $B = \{\text{all multiples of 4 from 4 to 65 inclusive}\}$

Find

$$|A \cap B|$$

3.1.3 Equal sets

Definition 8

Two sets A and B are *equal* iff the sets have identical elements and:

$$|A| = |B|$$

Neither order nor repetition matter.

3.1.4 Universal set

Definition 9

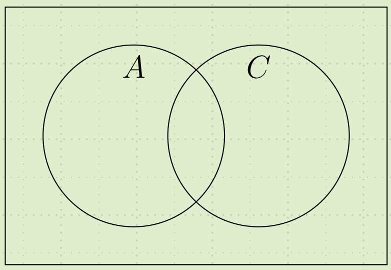
A *universal set* is the set of everything under discussion in a particular situation.

Example 12

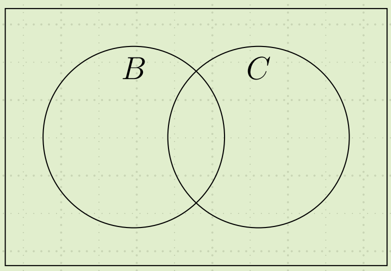
Let the universal set be $U = \{\text{whole numbers less than } 20\}$ and let

- $A = \{\text{squares less than } 20\}$
- $B = \{\text{even numbers less than } 20\}$
- $C = \{\text{odd squares less than } 20\}$

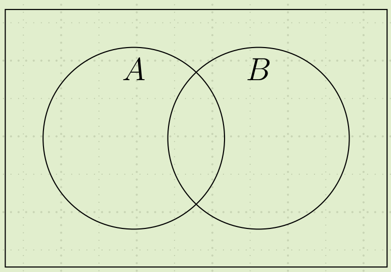
(a) Draw A and C on a Venn diagram, and place the numbers in the correct regions.



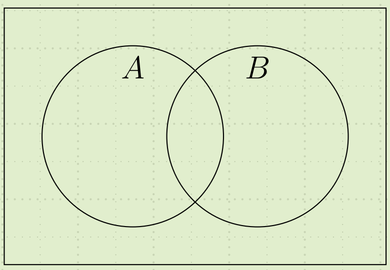
(b) Draw B and C on a Venn diagram, and place the numbers in the correct regions.



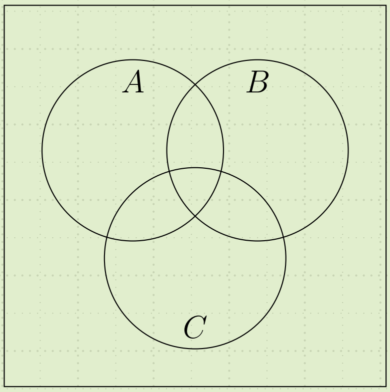
(c) Shade $A \cap B$ on a Venn diagram, and place the numbers in the correct regions.



(d) Shade $A \cup B$ on a Venn diagram, and place the numbers in the correct regions.



(e) Shade $(A \cap \overline{B}) \cup C$ on a Venn diagram, and place the numbers in the correct regions.

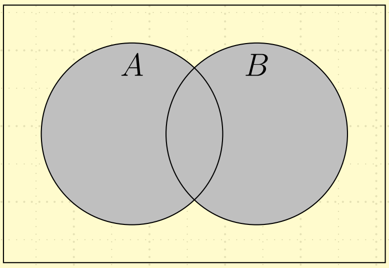


3.1.5 Counting rule

Theorem 3

For two sets A and B which are not disjoint,

$$|A \cup B| = |A| + |B| - |A \cap B|$$

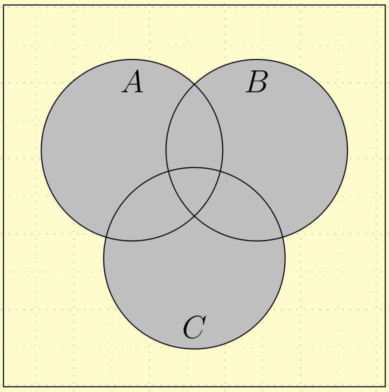


i.e. take out one instance of the duplicated values.

Theorem 4

For three sets A, B and C which are not disjoint,

$$|A \cup B \cup C| = |A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$



Further exercises

(A) **Ex 10C**
- Q9-18

(x1) **Ex 12C**
- Q9-22

3.2 Addition Rule

3.2.1 Mutually exclusive events

Theorem 5

If A and B are two mutually exclusive events, then

$$P(A \cup B) = P(A) + P(B)$$

Draw a Venn Diagram representing this situation.

List some other events which are mutually exclusive.

3.2.2 Not mutually exclusive

Theorem 6

If A and B are two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Draw a Venn Diagram representing this situation.

Example 13

[2014 CSSA 2U Q3] In a class of 30 girls, 13 are dancers and 23 are gymnasts. If 7 girls do both dance and gymnastics, what is the probability that a girl chosen at random does neither dance nor gymnastics?

(A) $\dfrac{1}{5}$ (B) $\dfrac{7}{30}$ (C) $\dfrac{1}{30}$ (D) $\dfrac{8}{15}$

Example 14

[1996 2U HSC Q8] Students studying at least one of the languages, French and Japanese, attend a meeting. Of the 28 students present, 18 study French and 22 study Japanese.

(i) What is the probability that a randomly chosen student studies French? **1**

(ii) What is the probability that two randomly chosen students both study French? **2**

(iii) What is the probability that a randomly chosen student studies both languages? **2**

Further exercises

Ⓐ **Ex 10D**
- Q3-13

(X1) **Ex 12D**
- Q7-13

Section 4

Multistage events

Learning Goal(s)

Knowledge
What multistage events are

Skills
Use the multiplication rule and tree diagrams to solve multistage problems

Understanding
How to identify multistage problems

By the end of this section am I able to:

13.11 Understand the concept of multi-stage experiments.

13.12 Use the multiplication law $P(A \cap B) = P(A)P(B)$ for independent events A and B and recognise the symmetry of independence in simple probability situations.

13.13 Construct probability tree diagrams

4.1 Independent events: product rule

Theorem 7

The *product rule* for an experiment with A_1, A_2, …, A_n as events:

$$P(A_1 A_2 A_3 \cdots A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n)$$

Important note

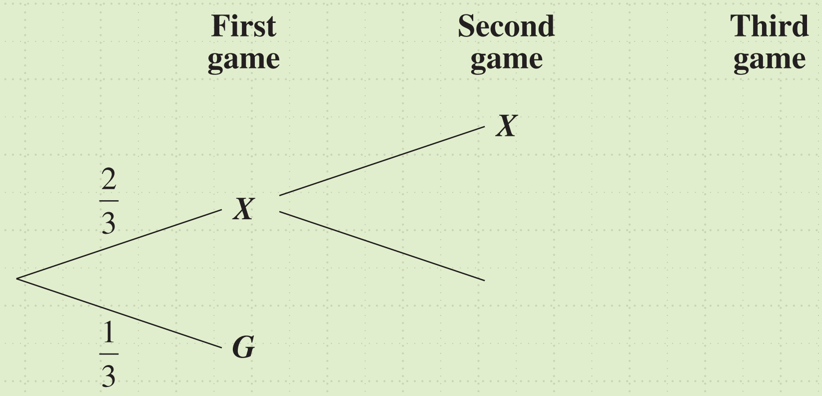
Be careful with your notation. Do not invent your own symbols!

Example 15

[**1998 2U HSC Q1**] A coin is tossed three times. What is the probability that 'heads' appears every time?

Example 16

[2008 2U HSC Q7] Xena and Gabrielle compete in a series of games. The series finishes when one player has won two games. In any game, the probability that Xena wins is $\frac{2}{3}$ and the probability that Gabrielle wins is $\frac{1}{3}$.
Part of a tree diagram for this series of games is shown.



(i) Complete the tree diagram showing the possible outcomes. **1**

(ii) What is the probability that Gabrielle wins the series? **2**

(iii) What is the probability that three games are played in the series? **2**

Answer: i. Show on diagram ii. $\frac{7}{27}$ iii. $\frac{4}{9}$

4.2 Dependent events

Important note

The ....sample........ ......space...... and ......event...... ......space...... will almost certainly change when proceeding from one stage to another.

Example 17

[2014 2U Q12] A packet of lollies contains 5 red lollies and 14 green lollies. Two lollies are selected at random without replacement.

(i) Draw a tree diagram to show the possible outcomes. Include the probability on each branch. **2**

(ii) What is the probability that the two lollies are of different colours? **1**

Answer: $\frac{70}{171}$

Example 18

[2011 2U HSC Q5] Kim has three red shirts and two yellow shirts. On each of the three days, Monday, Tuesday and Wednesday, she selects one shirt at random to wear. Kim wears each shirt that she selects only once.

(i) What is the probability that Kim wears a red shirt on Monday? **1**

(ii) What is the probability that Kim wears a shirt of the same colour on all three days? **1**

(iii) What is the probability that Kim does not wear a shirt of the same colour on consecutive days? **2**

Answer: (i) $\frac{3}{5}$ (ii) $\frac{1}{10}$ (iii) $\frac{3}{10}$

Further exercises

Ⓐ **Ex 10E**

- All questions

ⓍⒾ **Ex 12E**

- Q5-20

Section 5

Further tree diagrams

Important note

- Be careful with ....imbalanced.......... trees where some branches of a tree ......terminate...... ...early... .
- Questions which contain *eventually* may have a limiting sum appear.
- Tree diagram questions are the most common historically, in HSC papers.

5.1 Regular balanced trees

Example 19

[**2018 2U HSC Q14**] Two machines, A and B, produce pens. It is known that 10% of the pens produced by machine A are faulty and that 5% of the pens produced by machine B are faulty.

Answer: (a) 0.145 (b) 0.85625

(i) One pen is chosen at random from each machine. **1**

What is the probability that at least one of the pens is faulty?

(ii) A coin is tossed to select one of the two machines. Two pens are chosen at random from the selected machine. **2**

What is the probability that neither pen is faulty?

Example 20

[2012 NSGHS 2U Trial Q12] A bag contains one green, four blue and six red marbles.

(i) Two marbles are drawn from the bag with replacement. Find the probability that two blue marbles are drawn. **1**

(ii) What is the probability that at least one of the marbles is red or green? **2**

(iii) A single marble is now removed from the bag without noting its colour and it is replaced with a green marble. **3**

A marble is now drawn from the bag. What is the probability that it is green?

Answer: (i) $\frac{16}{121}$ (ii) $\frac{105}{121}$ (iii) $\frac{21}{121}$

5.2 Early terminating trees & multidisciplinary problems

Example 21

[**2016 2U HSC Q15**] An eight-sided die is marked with numbers 1, 2, ..., 8. A game is played by rolling the die until an 8 appears on the uppermost face. At this point the game ends.

Answer: (a) Show (b) $n = 12$

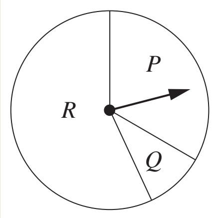
(i) Using a tree diagram, or otherwise, explain why the probability of the game ending before the fourth roll is **2**

$$\frac{1}{8} + \frac{7}{8} \times \frac{1}{8} + \left(\frac{7}{8}\right)^2 \times \frac{1}{8}$$

(ii) What is the smallest value of n for which the probability of the game ending before the n-th roll is more than $\frac{3}{4}$? **3**

Example 22

[2014 Ext 1 HSC Q14] Two players A and B play a game that consists of taking turns until a winner is determined. Each turn consists of spinning the arrow on a spinner once. The spinner has three sectors P, Q and R. The probabilities that the arrow stops in sectors P, Q and R are p, q and r respectively.



The rules of the game are as follows:

- If the arrow stops in sector P, then the player having the turn wins.
- If the arrow stops in sector Q, then the player having the turn loses and the other player wins.
- If the arrow stops in sector R, then the other player takes a turn.

Player A takes the first turn.

(i) Show that the probability of player A winning on the first or the second turn of the game is **2**

$$(1-r)(p+r)$$

(ii) Show that the probability that player A eventually wins the game is **3**

$$\frac{p+r}{1+r}$$

Example 23

[**2011 Ext 1 HSC**] A game is played by throwing darts at a target. A player can choose to throw two or three darts.

Darcy plays two games. In Game 1, he chooses to throw two darts, and wins if he hits the target at least once. In Game 2, he chooses to throw three darts, and wins if he hits the target at least twice.

The probability that Darcy hits the target on any throw is p, where $0 < p < 1$.

(i) Show that the probability that Darcy wins Game 1 is $2p - p^2$. **1**

(ii) Show that the probability that Darcy wins Game 2 is $3p^2 - 2p^3$. **1**

(iii) Prove that Darcy is more likely to win Game 1 than Game 2. **2**

(iv) Find the value of p for which Darcy is twice as likely to win Game 1 as he is to win Game 2. **2**

Answer: $p \approx 0.3596$

Further exercises

(A) Ex 10F

- Q3-19

(X1) Ex 12F

- Q8-22

Section 6

Conditional Probability

Learning Goal(s)

Knowledge
How to convert problem description into probability notation

Skills
Solve problems using the formula for conditional probability

Understanding
The origins of the formula for conditional probability and how to identify conditional probability problems from other types of probability problems

By the end of this section am I able to:

13.13 Understand the notion of conditional probability and recognise and use language that indicates conditionality.

13.14 Use the notation $P(A|B)$ and the formula $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$, $P(B) \neq 0$ for conditional probability.

13.15 Understand the notion of independence of an event A from an event B , as defined by $P(A|B) = P(A)$.

Important note

Ⓛ Keyword: *given*

- Consequently, the ....sample.... space and ....event.... space shrinks.

6.1 Definitions

Definition 10

The *conditional probability* of A occurring given B has occurred, is denoted $P(A|B)$.

$$P(A|B) = \frac{|\text{reduced event space}|}{|\text{reduced sample space}|} = \frac{|A \cap B|}{|B|}$$

Derive alternative formula for $P(A|B)$ by dividing throughout by $|S|$:

$$\begin{aligned} P(A|B) &= \frac{|A \cap B|}{|B|} \\ &= \frac{\frac{|A \cap B|}{|S|}}{\frac{|B|}{|S|}} \\ &= \frac{P(A \cap B)}{P(B)} \end{aligned}$$

Steps

1. Remove all elements not in B from the sample space.
2. Remove all elements not in B from the event space A.
3. Compute.

Example 24

In a certain population, 35% have blue eyes, 15% have blond hair, and 10% have blue eyes and blond hair. A person is chosen from this population at random.

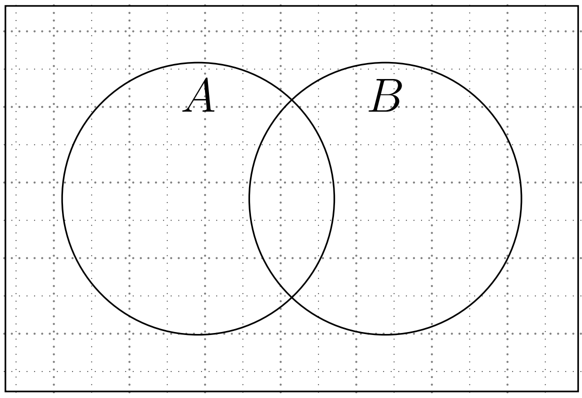
(a) Find the probability that they have blond hair, given that they have blue eyes.

(b) Find the probability that they have blue eyes, given that they have blond hair.

Answer: (a) 0.29 (b) 0.67 (to 2 dp)

Apply the formula Let $A = \{\text{blue eyed population}\}$ and $B = \{\text{blond haired population}\}$

Verify with a Venn diagram:



Example 25

[2018 VCE Mathematical Methods Paper 1 Q6] Two boxes each contain four stones that differ only in colour.

- Box 1 contains four black stones.
- Box 2 contains two black stones and two white stones.

A box is chosen randomly and one stone is drawn randomly from it. Each box is equally likely to be chosen, as is each stone.

(a) What is the probability that the randomly drawn stone is black? **2**

(b) It is not known from which box the stone has been drawn. **2**

Given that the stone that is drawn is black, what is the probability that it was drawn from Box 1?

Answer: (a) $\frac{3}{4}$ (b) $\frac{2}{3}$

Example 26

[2017 VCE Mathematical Methods Paper 1 Q8] For events A and B from a sample space, $P(A|B) = \dfrac{1}{5}$ and $P(B|A) = \dfrac{1}{4}$. Let $P(A \cap B) = p$.

(a) Find $P(A)$ in terms of p. **1**

(b) Find $P(\overline{A} \cap \overline{B})$ in terms of p. **2**

(c) Given that $P(A \cup B) \leq \frac{1}{5}$, state the largest possible interval for p. **2**

Answer: (a) $4p$ (b) $1 - 8p$ (c) $p \in \left(0, \frac{1}{40}\right]$

Example 27

[2015 VCE Mathematical Methods (CAS) Paper 1 Q5] An egg marketing company buys its eggs from farm A and farm B. Let p be the proportion of eggs that the company buys from farm A. The rest of the company's eggs come from farm B. Each day, the eggs from both farms are taken to the company's warehouse.

Assume that $\frac{3}{5}$ of all eggs from farm A have white eggshells and $\frac{1}{5}$ of all eggs from farm B have white eggshells.

(a) An egg is selected at random from the set of all eggs at the warehouse. Find, in terms of p, the probability that the egg has a white eggshell **1**

(b) Another egg is selected at random from the set of all eggs at the warehouse.

i. Given that the egg has a white eggshell, find, in terms of p, the probability that it came from farm B. **2**

ii. If the probability that this egg came from farm B is 0.3, find the value of p. **1**

Answer: (a) $\frac{2p+1}{5}$ (b) i. $\frac{1-p}{2p+1}$ ii. $\frac{7}{16}$

Example 28

[2013 James Ruse Ext 1 Trial Q10] Two year 12 students are to be randomly selected from a pool of N year 12 students, n of whom are from James Ruse. If it is known that at least one student is from James Ruse, what is the chance that both students are from James Ruse?

(A) $\dfrac{n-1}{2N-n-1}$ (B) $\dfrac{n-1}{2N+n+1}$ (C) $\dfrac{n-1}{2N+n-1}$ (D) $\dfrac{n-1}{2N-n+1}$

6.2 Revisiting independent events

Definition 11

Two events A and B are *independent* if

$$P(A|B) = P(A)$$

Example 29

Two dice are thrown one after the other.

- Let A be the event *the first die is odd.*
- Let B be the event *the second die is 1, 2 or 3.*
- Let C be the event *the sum is five*

Which of the three pairs of events are independent? **Answer:** A and B, A and C.

Further exercises

Ⓐ **Ex 10G**

- All questions

ⓍⅠ **Ex 12G**

- All questions

NESA Reference Sheet – calculus based courses

NSW Education Standards Authority

2020 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Advanced
Mathematics Extension 1
Mathematics Extension 2

REFERENCE SHEET

Measurement

Length

$$l = \frac{\theta}{360} \times 2\pi r$$

Area

$$A = \frac{\theta}{360} \times \pi r^2$$

$$A = \frac{h}{2}(a + b)$$

Surface area

$$A = 2\pi r^2 + 2\pi rh$$

$$A = 4\pi r^2$$

Volume

$$V = \frac{1}{3}Ah$$

$$V = \frac{4}{3}\pi r^3$$

Functions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For $ax^3 + bx^2 + cx + d = 0$:

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

and $\alpha\beta\gamma = -\frac{d}{a}$

Relations

$$(x - h)^2 + (y - k)^2 = r^2$$

Financial Mathematics

$$A = P(1 + r)^n$$

Sequences and series

$$T_n = a + (n - 1)d$$

$$S_n = \frac{n}{2}[2a + (n - 1)d] = \frac{n}{2}(a + l)$$

$$T_n = ar^{n-1}$$

$$S_n = \frac{a(1 - r^n)}{1 - r} = \frac{a(r^n - 1)}{r - 1},\ r \neq 1$$

$$S = \frac{a}{1 - r},\ |r| < 1$$

Logarithmic and Exponential Functions

$$\log_a a^x = x = a^{\log_a x}$$

$$\log_a x = \frac{\log_b x}{\log_b a}$$

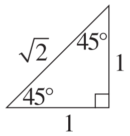
$$a^x = e^{x \ln a}$$

Trigonometric Functions

$$\sin A = \frac{\text{opp}}{\text{hyp}}, \quad \cos A = \frac{\text{adj}}{\text{hyp}}, \quad \tan A = \frac{\text{opp}}{\text{adj}}$$

$$A = \frac{1}{2}ab\sin C$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

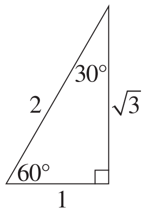


$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$l = r\theta$$

$$A = \frac{1}{2}r^2\theta$$



Trigonometric identities

$$\sec A = \frac{1}{\cos A}, \ \cos A \neq 0$$

$$\operatorname{cosec} A = \frac{1}{\sin A}, \ \sin A \neq 0$$

$$\cot A = \frac{\cos A}{\sin A}, \ \sin A \neq 0$$

$$\cos^2 x + \sin^2 x = 1$$

Compound angles

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

If $t = \tan\frac{A}{2}$ then $\sin A = \frac{2t}{1+t^2}$

$$\cos A = \frac{1-t^2}{1+t^2}$$

$$\tan A = \frac{2t}{1-t^2}$$

$$\cos A\cos B = \frac{1}{2}\left[\cos(A - B) + \cos(A + B)\right]$$

$$\sin A\sin B = \frac{1}{2}\left[\cos(A - B) - \cos(A + B)\right]$$

$$\sin A\cos B = \frac{1}{2}\left[\sin(A + B) + \sin(A - B)\right]$$

$$\cos A\sin B = \frac{1}{2}\left[\sin(A + B) - \sin(A - B)\right]$$

$$\sin^2 nx = \frac{1}{2}(1 - \cos 2nx)$$

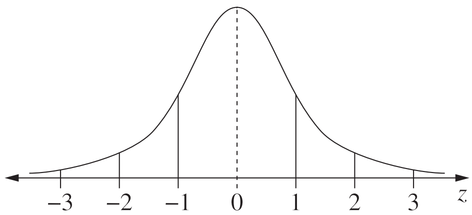
$$\cos^2 nx = \frac{1}{2}(1 + \cos 2nx)$$

Statistical Analysis

$$z = \frac{x - \mu}{\sigma}$$

An outlier is a score
less than $Q_1 - 1.5 \times IQR$
or
more than $Q_3 + 1.5 \times IQR$

Normal distribution



- approximately 68% of scores have z-scores between –1 and 1
- approximately 95% of scores have z-scores between –2 and 2
- approximately 99.7% of scores have z-scores between –3 and 3

$$E(X) = \mu$$

$$\operatorname{Var}(X) = E\left[(X - \mu)^2\right] = E\left(X^2\right) - \mu^2$$

Probability

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \ P(B) \neq 0$$

Continuous random variables

$$P(X \le x) = \int_a^x f(x)\,dx$$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

Binomial distribution

$$P(X = r) = {}^nC_r p^r (1 - p)^{n-r}$$

$$X \sim \operatorname{Bin}(n, p)$$

$$\Rightarrow \quad P(X = x)$$

$$= \binom{n}{x} p^x (1 - p)^{n-x}, \ x = 0, 1, \ldots, n$$

$$E(X) = np$$

$$\operatorname{Var}(X) = np(1 - p)$$

Differential Calculus

Function	Derivative
$y = f(x)^n$	$\frac{dy}{dx} = nf'(x)\left[f(x)\right]^{n-1}$
$y = uv$	$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$
$y = g(u)$ where $u = f(x)$	$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$
$y = \frac{u}{v}$	$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
$y = \sin f(x)$	$\frac{dy}{dx} = f'(x)\cos f(x)$
$y = \cos f(x)$	$\frac{dy}{dx} = -f'(x)\sin f(x)$
$y = \tan f(x)$	$\frac{dy}{dx} = f'(x)\sec^2 f(x)$
$y = e^{f(x)}$	$\frac{dy}{dx} = f'(x)e^{f(x)}$
$y = \ln f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{f(x)}$
$y = a^{f(x)}$	$\frac{dy}{dx} = (\ln a)f'(x)a^{f(x)}$
$y = \log_a f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{(\ln a)f(x)}$
$y = \sin^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \cos^{-1} f(x)$	$\frac{dy}{dx} = -\frac{f'(x)}{\sqrt{1-\left[f(x)\right]^2}}$
$y = \tan^{-1} f(x)$	$\frac{dy}{dx} = \frac{f'(x)}{1+\left[f(x)\right]^2}$

Integral Calculus

$$\int f'(x)\left[f(x)\right]^n dx = \frac{1}{n+1}\left[f(x)\right]^{n+1} + c$$

where $n \neq -1$

$$\int f'(x)\sin f(x)\,dx = -\cos f(x) + c$$

$$\int f'(x)\cos f(x)\,dx = \sin f(x) + c$$

$$\int f'(x)\sec^2 f(x)\,dx = \tan f(x) + c$$

$$\int f'(x)e^{f(x)}\,dx = e^{f(x)} + c$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln\left|f(x)\right| + c$$

$$\int f'(x)a^{f(x)}\,dx = \frac{a^{f(x)}}{\ln a} + c$$

$$\int \frac{f'(x)}{\sqrt{a^2 - \left[f(x)\right]^2}}\,dx = \sin^{-1}\frac{f(x)}{a} + c$$

$$\int \frac{f'(x)}{a^2 + \left[f(x)\right]^2}\,dx = \frac{1}{a}\tan^{-1}\frac{f(x)}{a} + c$$

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

$$\int_a^b f(x)\,dx$$

$$\approx \frac{b-a}{2n}\left\{f(a) + f(b) + 2\left[f\left(x_1\right) + \cdots + f\left(x_{n-1}\right)\right]\right\}$$

where $a = x_0$ and $b = x_n$

Combinatorics

$${}^nP_r = \frac{n!}{(n-r)!}$$

$$\binom{n}{r} = {}^nC_r = \frac{n!}{r!\,(n-r)!}$$

$$(x+a)^n = x^n + \binom{n}{1}x^{n-1}a + \cdots + \binom{n}{r}x^{n-r}a^r + \cdots + a^n$$

Vectors

$$\left|\underset{\sim}{u}\right| = \left|x\underset{\sim}{i} + y\underset{\sim}{j}\right| = \sqrt{x^2+y^2}$$

$$\underset{\sim}{u} \cdot \underset{\sim}{v} = \left|\underset{\sim}{u}\right|\left|\underset{\sim}{v}\right|\cos\theta = x_1x_2 + y_1y_2,$$

where $\underset{\sim}{u} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$

and $\underset{\sim}{v} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$

$$\underset{\sim}{r} = \underset{\sim}{a} + \lambda\underset{\sim}{b}$$

Complex Numbers

$$z = a + ib = r(\cos\theta + i\sin\theta)$$
$$= re^{i\theta}$$

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$
$$= r^n e^{in\theta}$$

Mechanics

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$x = a\cos(nt + \alpha) + c$$

$$x = a\sin(nt + \alpha) + c$$

$$\ddot{x} = -n^2(x - c)$$

References

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019a). *CambridgeMATHS Stage 6 Mathematics Advanced Year 11* (1st ed.). Cambridge Education.

Pender, W., Sadler, D., Ward, D., Dorofaeff, B., & Shea, J. (2019b). *CambridgeMATHS Stage 6 Mathematics Extension 1 Year 11* (1st ed.). Cambridge Education.